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Asynchronous domain decomposition methods for continuous casting problem

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Abstract

Two asynchronous domain decomposition methods (which appear to be a two-stage Schwarz alternating algorithms) to solve the finite difference schemes approximating dynamic continuous casting problem are theoretically and numerically studied. Fully implicit and semi-implicit (implicit for the diffusion operator while explicit for the nonlinear convective term) finite difference schemes are considered. Unique solvability of the finite difference schemes as well as a monotone dependence of the solution on the right-hand side (the so-called comparison theorem) are proved. Geometric rate of convergence for the iterative methods is investigated, the comparison theorem being the main tool of this study. Numerical results are included and analyzed.
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1. Introduction

The general idea of the Schwarz alternating methods is to solve the boundary-value problem restricted to each subdomain, using as the boundary conditions the function values of the approximative solution of the neighboring subdomains. One of the advantages of the additive Schwarz is that the solutions in the subdomains can be handled by different processors of a parallel computer. However, due to the mutual waits among the processors when a synchronous method is applied, it leads to high loss of calculating time. To exploit the asynchronous parallel computing capacity of a

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multi-processor system, we propose and investigate theoretically and numerically the asynchronous algorithms for solving "highly" nonlinear finite-dimensional problem.

Algorithm 1 (ASM1).

1. Divide the domain of a boundary-value problem into p overlapping subdomains and construct approximative subproblems in these subdomains.
2. Solve simultaneously the subproblems in the slave processors.
3. When the local stopping criterion in a slave processor is reached, send information about this to the master processor and keep calculating further.
4. When all slaves have finished the calculations, send the subsolutions to the master processor for updating the information for all slave processors.
5. If the accuracy is reached, then **STOP**, else goto 2.

Algorithm 2 (ASM2).

1. Divide the domain of a boundary-value problem into p overlapping subdomains and construct approximative subproblems in these subdomains.
2. Solve simultaneously the subproblems in the slave processors.
3. When the local stopping criterion in a slave processor is reached, send subsolution to the master processor and check if there is a new information from the neighboring subdomains. If yes, then update it and restart the calculations, otherwise keep calculating further.
4. When all slaves have finished the calculations, send the subsolutions to the master processor for updating the information for all slave processors.
5. If the accuracy is reached, then **STOP**, else goto 2.

In **Algorithm 1** we do not use the newest available information. This slows the convergence. However, it is much faster to just send sign to the master that the processor is ready than send the whole subsolution.

In **Algorithm 2** we send the subsolution to the master whenever it is gained. This increases the total calculation time. On the other hand, we use the newest available information which decreases the calculation time.

Intuitively, if there is a large load imbalance, i.e., if some processors have substantially more work than others, one can expect the asynchronous versions to converge faster than the synchronous one. It is also expected that ASM2 would be faster than ASM1. We will discuss these questions in more detail in Sections 5 and 6.

Stefan problem with prescribed convection, whose special case is the continuous casting problem, has been considered in a number of articles (cf., e.g., [11,20–22], and references therein), where the existence and uniqueness of a weak solution has been proved.

A numerical schemes for this problem, based on an implicit discretization of the diffusion term and an explicit discretization of the nonlinear convection term (semi-implicit scheme in our terminology), has been studied in [7–9]. A fully implicit approximation of the problem has been considered in [15], where the existence and uniqueness of the solution for the mesh scheme, as well as the convergence of the different iterative methods, has been proved.

A parallel algorithms for the systems of nonlinear algebraic equations (mesh schemes for the classical Stefan problem are the examples of such systems) have been considered in [1,2,23]. In the recent article [3], the geometric convergence of a class of asynchronous iterative methods for the system of nonlinear algebraic equations has been proved. A number of articles deal with parallel solution of variational inequalities (cf. [5,13,14,17,24–27]), where they study the so-called obstacle problem and its generalization. In our notations, it is a partial case of problem (4) with diagonal matrix B.

2. Continuous casting problem

A continuous casting problem can be stated mathematically as follows. Let $\Omega=\{0<x_1<L_{x_1}, 0<x_2<L_{x_2}\}$ be the rectangular domain with the boundary $\Gamma=\partial\Omega$ consisting of two parts: $\Gamma_1=\{x\in\partial\Omega : x_2=0 \vee x_2=L_{x_2}\}$ and $\Gamma_2=\{x\in\partial\Omega\backslash\Gamma_1\}$. We assume that the domain $\Omega\subset\mathbb{R}^2$ is occupied by a thermodynamically homogeneous and isotropic steel. We denote by $H(x,t)$ the enthalpy related to the unit mass and by $u(x,t)$ the temperature for $(x,t)\in\Omega\times]0,T[$. We have constitutive law

$$H=H(u)=\rho\int_0^u c(\Theta)\,\mathrm{d}\Theta+\rho L(1-f_{\mathrm{s}}(u)) \text{ in } \Omega\times]0,T[,$$

where ρ is the density, $c(u)$ is the specific heat, L is the latent heat and $f_{\mathrm{s}}(u)$ is the solid fraction.

In a steel casting process the enthalpy $H(u)$ is an increasing function $\mathbb{R}\to\mathbb{R}$, involving near vertical segments. They correspond to phase transition states [18]. On the other hand, when a copper casting problem is studied, graph $H(u)$ has a vertical segment for $u=T_L=T_{\mathrm{S}}$ [12]. Further, we include in our consideration both problems and denote by $H(u)$, $u\in\mathbb{R}$, a maximal monotone, generally multivalued, graph.

We also suppose that graph $H(u)$ is uniformly monotone: there exists a positive constant α such that

$$(\gamma_1-\gamma_2,u_1-u_2)\geqslant\alpha(u_1-u_2,u_1-u_2)\quad\forall u_1,u_2\ \forall\gamma_i\in H(u_i). \tag{1}$$

Now a continuous casting process can be described by a boundary-value problem, formally written in the following pointwise form: find $u(x,t)$ and $\gamma(x,t)$ such that

$$(\mathrm{P})\begin{cases}\dfrac{\partial\gamma}{\partial t}+v\dfrac{\partial\gamma}{\partial x_2}-\Delta u=0 \quad \text{for } x\in\Omega,\ t>0,\\ u=z(x_1,t) \quad \text{for } x\in\Gamma_1,\ t>0,\\ \dfrac{\partial u}{\partial n}+au+b|u|^3u=g, \quad a\geqslant 0, \quad b\geqslant 0 \quad \text{for } x\in\Gamma_2,\ t>0,\\ \gamma=H_0(x) \quad \text{for } x\in\bar{\Omega},\ t=0,\\ \gamma(x,t)\in H(u(x,t)) \quad \text{for } x\in\Omega,\ t>0.\end{cases}$$

Below we suppose that the boundary temperature $z(x_1,t)$ at any point of Γ_1, for all $t\geqslant 0$, does not coincide with the phase transition temperature $T_L=T_{\mathrm{S}}$. In other words, enthalpy function H has a single value at all these points. This corresponds to physical meaning of the problem, because the incoming material (points $x\in\Gamma_1{:}\,x_2=0$) is in liquid state, while the outcoming material (points $x\in\Gamma_1{:}\,x_2=L_{x_2}$) is in solid state.

The existence and uniqueness of a weak solution for problem (P) are proved in [20].

We approximate the problem (P) by an implicit in time, finite difference scheme and by a semi-implicit finite difference scheme, using for the approximation in space variables, a finite element method with the quadrature rules.

Let T_h be a triangulation of Ω in the rectangular elements δ of dimensions $h_1 \times h_2$ and $V_h = \{u_h(x) \in H^1(\Omega) : u_h(x) \in Q_1$ for all $\delta \in T_h\}$, where Q_1 is the space of bilinear functions. By $\Pi_h v(x)$ we denote the V_h-interpolant of a continuous function $v(x)$, i.e., $\Pi_h v(x) \in V_h$ and coincides with $v(x)$ in the mesh nodes—vertices of all $\delta \in T_h$. We also use an interpolation operator P_h, which is defined as follows: for any continuous function $v(x)$ the function $P_h v(x)$ is piecewise linear in x_1, piecewise constant in x_2 and on $\delta = [x_1, x_1 + h_1] \times [x_2, x_2 + h_2]$ it coincides with $v(x)$ at $(x_1, x_2 + h_2)$ and $(x_1 + h_1, x_2 + h_2)$.

Let further $V_h^0 = \{u_h(x) \in V_h : u_h(x) = 0$ for all $x \in \Gamma_1\}$, $V_h^z = \{u_h(x) \in V_h : u_h(x) = z_h$ for all $x \in \Gamma_1\}$. Here z_h is the bilinear interpolation of z on the boundary Γ_1. For any continuous function $v(x)$ we define the quadrature formulas

$$S_\delta(v) = \int_\delta \Pi_h v \, \mathrm{d}x, \quad S_\Omega(v) = \sum_{\delta \in T_h} S_\delta v,$$

$$S_{\partial\delta}(v) = \int_{\partial\delta} \Pi_h v \, \mathrm{d}x, \quad S_{\Gamma_2}(v) = \sum_{\partial\delta \in T_h \cap \bar{\Gamma} 2} S_{\partial\delta}(v),$$

$$E_\delta(v) = \int_\delta P_h v \, \mathrm{d}x, \quad E_\Omega(v) = \sum_{\delta \in T_h} E_\delta(v).$$

Let also $\omega_\tau = \{t_k = k\tau,\ 0 \leqslant k \leqslant M,\ M\tau = T\}$ be an uniform mesh in time on the segment $[0, T]$ and

$$\partial_{\bar{t}} \gamma = \frac{1}{\tau}(\gamma(x,t) - \gamma(x, t-\tau)).$$

Then the implicit in time finite difference scheme with up-wind approximation of the convective term $v\partial\gamma/\partial x_2$ can be written as follows: for all $t \in \omega_\tau$, $t > 0$, find $u_h \in V_h^z$ and $\gamma_h \in V_h$ such that

$$\begin{aligned} & S_\Omega(\partial_{\bar{t}} \gamma_h \eta_h) + E_\Omega\left(v(t)\frac{\partial \gamma_h}{\partial x_2}\eta_h\right) + S_\Omega(\nabla u_h \nabla \eta_h) \\ & \quad + S_{\Gamma_2}((a u_h + b|u_h|^3 u_h)\eta_h) = S_{\Gamma_2}(g\eta_h) \quad \text{for all } \eta_h \in V_h^0, \\ & \gamma_h(x,t) \in H(u_h(x,t)) \quad \text{for all mesh nodes } x. \end{aligned} \tag{2}$$

When constructing the semi-implicit mesh scheme, we approximate the term $(\partial/\partial t + v(t)\partial/\partial x_2)\gamma$ by using the characteristics of the first-order differential operator (similar to [7,10]). Namely, if (x_1, x_2, t) is the mesh point on the time level t we choose $\tilde{x}_2 = x_2 - \int_{t-\tau}^t v(\xi)\,\mathrm{d}\xi$ and approximate this term by

$$\left(\frac{\partial}{\partial t} + v(t)\frac{\partial}{\partial x_2}\right)\gamma \approx \frac{1}{\tau}(\gamma(x_1, x_2, t) - \gamma(x_1, \tilde{x}_2, t-\tau)).$$

If $\tilde{x}_2 < 0$ we then put

$$\tilde{\gamma}(x, t-\tau) = \gamma(x_1, 0, t-\tau).$$

Note that $\gamma(x_1, 0, t-\tau) = H(z(x_1, t-\tau))$ with single values $H(z(x_1, t-\tau))$ of H at these points, as it was mentioned above.

In what follows we use the notation

$$d_{\bar{t}}\gamma = \frac{1}{\tau}(\gamma(x,t) - \tilde{\gamma}(x,t-\tau))$$

for the difference quotient in each mesh point on time level t.

Now, the semi-implicit finite difference scheme for problem (P) is: for all $t \in \omega_\tau$, $t > 0$, find $u_h \in V_h^z$ and $\gamma_h \in V_h$ such that

$$\begin{aligned}
&S_\Omega(d_{\bar{t}}\gamma_h \eta_h) + S_\Omega(\nabla u_h \nabla \eta_h) + S_{\Gamma_2}((a u_h + b|u_h|^3 u_h)\eta_h) \\
&\quad = S_{\Gamma_2}(g\eta_h) \quad \text{for all } \eta_h \in V_h^0, \\
&\gamma_h(x,t) \in H(u_h(x,t)) \quad \text{for all mesh nodes } x.
\end{aligned} \tag{3}$$

Let $N_0 = \operatorname{card} V_h^0$ and $u \in \mathbb{R}^{N_0}$ be the vector of nodal values for $u_h \in V_h^0$. We use the notation $u_h \Leftrightarrow u$ for this bijection. We define $N_0 \times N_0$ matrices A and B and nonlinear operator C by the following relations: for all $V_h^0 \ni u_h \Leftrightarrow u \in \mathbb{R}^{N_0}$ and $V_h^0 \ni \eta_h \Leftrightarrow \eta \in \mathbb{R}^{N_0}$

$$\begin{aligned}
&(Au, \eta) = S_\Omega(\nabla u_h \nabla \eta_h) + S_{\Gamma_2}(a u_h \eta_h), \\
&(Bu, \eta) = S_\Omega\left(\frac{1}{\tau} u_h \eta_h\right) + E_\Omega\left(v(t)\frac{\partial u_h}{\partial x_2}\eta_h\right), \\
&(Cu, \eta) = S_{\Gamma_2}(b|u_h|^3 u_h \eta_h).
\end{aligned}$$

Further, we define a vector f: $(f, \eta) = S_{\Gamma_2}(g\eta_h) + S_\Omega((1/\tau)H(u_h(x, t-\tau))\eta_h)$. Let now $\tilde{z}_h(x) \in V_h$ be the function which is equal to z_h on $\bar{\Gamma}_1$ and 0 for all nodes in $\Omega \cup \Gamma_2$. Then f_0 is defined by the equality

$$(f_0, \eta) = S_\Omega(\nabla \tilde{z}_h, \nabla \eta_h) + E_\Omega\left(v(t)\frac{\partial \Pi_h(H(\tilde{z}_h))}{\partial x_2}\eta_h\right) \quad \text{for all } \eta_h \in V_h^0.$$

(Here we use again the fact that the graph $H(u)$ is single valued for $u = \tilde{z}_h(x)$, when x is a mesh point). Finally, we get $F = f - f_0$.

In these notations, the algebraic form for the implicit mesh scheme (2) at fixed time level is

$$Au + B\gamma + Cu = F, \quad \gamma \in H(u). \tag{4}$$

If we set

$$(Bu, \eta) = S_\Omega\left(\frac{1}{\tau} u_h \eta_h\right)$$

and

$$(f, \eta) = S_{\Gamma_2}(g\eta_h) + S_\Omega\left(\frac{1}{\tau}\tilde{H}_h \eta_h\right),$$

then the semi-implicit mesh scheme (3) has also the algebraic form (4).

The proof of the following lemma is straightforward:

Lemma 1. *The matrices A, B and the operators C, H have the following properties*:

$$A \text{ and } B \text{ are } M\text{-matrices}, \tag{5}$$

A is weakly diagonally dominant in columns:

$$\sum_{j \neq i}^{N_0} |a_{ji}|/a_{ii} \leqslant 1 \ \forall i, \tag{6}$$

B is strictly diagonally dominant in columns

$$\sum_{j \neq i}^{N_0} |b_{ji}|/b_{ii} \leqslant \beta < 1 \ \forall i, \tag{7}$$

(*in fact*, *for the semi-implicit scheme matrix B is diagonal*) *operators H and C have the diagonal forms*

$$H(u) = (H(u_1), H(u_2), \ldots, H(u_{N_0}))^{\mathrm{t}}, \quad Cu = (c_1(u_1), c_2(u_2), \ldots, c_N(u_{N_0}))^{\mathrm{t}}, \tag{8}$$

where c_i are continuous nondecreasing functions and $H(.)$ is maximal monotone and uniformly monotone graph (*see* (1)).

Note, that $\beta = \tau/(\tau + h_2)$ for the case of implicit finite difference scheme, while $\beta = 0$ for the semi-implicit scheme.

Below we use the following notations:

$$u \geqslant 0 \Leftrightarrow u_i \geqslant 0 \ \forall i, \quad A \geqslant 0 \Leftrightarrow a_{ij} \geqslant 0 \ \forall i, j.$$

It is easy to prove that there exist a subsolution $(\underline{u}, \underline{\gamma})$

$$A\underline{u} + B\underline{\gamma}C\underline{u} \leqslant F, \quad \underline{\gamma} \in H(\underline{u}) \tag{9}$$

and a supersolution $(\bar{u}, \bar{\gamma})$

$$A\bar{u} + B\bar{\gamma}C\bar{u} \geqslant F, \quad \bar{\gamma} \in H(\bar{u}) \tag{10}$$

for problem (4).

In fact, to construct a supersolution we choose a vector $\bar{u}$ with coordinate $\bar{u}_i \equiv c_1 = \text{const}$, corresponding to the nodes in $\Omega \backslash \bar{\Gamma}_1$, while $\bar{u}_i = z_i$ for the coordinates, corresponding to the nodes in $\bar{\Gamma}_1$. If c_1 is sufficiently big and $\bar{\gamma}_{\mathrm{i}} \in H(\bar{u}_i)$ then the pair $(\bar{u}, \bar{\gamma})$ is a supersolution for problem (4). Similarly, if $\underline{u}$ has coordinates $\underline{u}_i \equiv c_2 = \text{const}$, corresponding to the nodes in $\Omega \backslash \bar{\Gamma}_1$, while $\bar{u}_i = z_i$ for the coordinates corresponding to the nodes in $\bar{\Gamma}_1$ with sufficiently small c_2 and $\underline{\gamma}_i \in H(\underline{u}_i)$, then the pair $(\underline{u}, \underline{\gamma})$ is a subsolution for problem (4).

3. Existence and uniqueness of the solution, comparison theorem

The result of the theorem below follows from [16], but for the convenience of reading we give a sketch of the proof.

Theorem 1. *Let $\mathscr{A}, \mathscr{B}$ be $N \times N$ M-matrices, $\mathscr{A}$ has weak diagonal dominance in columns while $\mathscr{B}$ is strictly diagonally dominant in columns. Let further C and H be diagonal maximal monotone operators in $\mathbb{R}^N$, C be continuous, and let the problem*

$$\mathscr{A}u + \mathscr{B}\gamma + Cu = F, \quad \gamma \in H(u) \tag{11}$$

have a subsolution $(\underline{u}, \underline{\gamma})$ and a supersolution $(\bar{u}, \bar{\gamma})$:

$$\mathscr{A}\underline{u} + \mathscr{B}\underline{\gamma}C\underline{u} \leqslant F \leqslant \mathscr{A}\bar{u} + \mathscr{B}\bar{\gamma}C\bar{u},$$

$$\underline{\gamma} \in H(\underline{u}), \quad \bar{\gamma} \in H(\bar{u}).$$

Then

(1) *problem* (11) *has a unique solution (u, γ) for any $F \in \mathbb{R}^{\mathrm{N}}$ and*
(2) *if (u^1, γ^1) and (u^2, γ^2) correspond to the right-hand sides F^1 and F^2, then the inequality $F^1 \geqslant F^2$ implies*

$$(u^1, \gamma^1) \geqslant (u^2, \gamma^2).$$

Proof. Let $\mathscr{A}^0 = \mathrm{diag}(\mathscr{A}), \mathscr{B}^0 = \mathrm{diag}(\mathscr{B})$ and $\mathscr{A}^1 = \mathscr{A} - \mathscr{A}^0, \mathscr{B}^1 = \mathscr{B} - \mathscr{B}^0$.

We consider the auxiliary problem

$$\mathscr{A}^0 u + \mathscr{B}^0\gamma + Cu = F - \mathscr{A}^1 v - \mathscr{B}^1\eta, \quad \gamma \in H(u) \tag{12}$$

for any fixed (v, η) from the ordered interval $\langle(\underline{u}, \underline{\gamma}), (\bar{u}, \bar{\gamma})\rangle$.

As the operator $\mathscr{A}^0 + \mathscr{B}^0 \circ H + C$ is strictly maximal monotone and coercive due to the positive definiteness of $\mathscr{A}^0$, there exists an unique solution $u = u(v, \eta)$ of problem (12) (cf. [6,19]). The component $\gamma = \gamma(v, \eta)$ of the solution for auxiliary problem (12) is also defined uniquely:

$$\gamma = (\mathscr{B}^0)^{-1}(F - \mathscr{A}^1 v - \mathscr{B}^1\eta - \mathscr{A}^0 u - Cu).$$

We define an operator $\mathscr{P}$ by the equality $\mathscr{P}(v, \eta) \equiv (u(v, \eta), \gamma(u, \eta))$.

Owing to the inequalities $\mathscr{A}^1 \leqslant 0$, $\mathscr{B}^1 \leqslant 0$ and to the definition of a supersolution we have

$$\mathscr{A}^0 u + \mathscr{B}^0\gamma + Cu(v) = F - \mathscr{A}^1 v - \mathscr{B}^1\eta \leqslant F - \mathscr{A}^1\bar{u} - \mathscr{B}^1\bar{\gamma} \leqslant \mathscr{A}^0\bar{u} + \mathscr{B}^0\bar{\gamma} + C\bar{u},$$

which implies $(u, \gamma) \leqslant (\bar{u}, \bar{\gamma})$. Similarly, $(u, \gamma) \geqslant (\underline{u}, \underline{\gamma})$. It means, that the operator $\mathscr{P}$ maps the ordered interval $\langle(\underline{u}, \underline{\gamma}), (\bar{u}, \bar{\gamma})\rangle$ into itself.

Using similar arguments, it is easy to prove that $\mathscr{P}$ is monotone: if $(v^1, \eta^1) \leqslant (v^2, \eta^2)$ then $(u(v^1), \gamma(\eta^1)) \leqslant (u(v^2), \gamma(\eta^2))$. Now, Kolodner–Tartar theorem [4] ensures the existence of a fixed point to operator $\mathscr{P}$. Obviously, this fixed point is a solution of problem (11).

Let now $F^1 \geqslant F^2$ and (u^1, γ^1), (u^2, γ^2) be the corresponding solutions for problem (11). We use the following notations for a subsets of indices: $I_- = \{i : u_i^1 < u_i^2\}$; $J_- = \{i : \gamma_i^1 < \gamma_i^2\}$, $M = I_- \cup J_-$. Let the coordinates of a vector η be defined by

$$\eta_i = \{1 \text{ for } i \in M; 0 \text{ for } i \notin M\}. \tag{13}$$

We note, that

$$u_i^1 \leqslant u_i^2,\ \gamma_i^1 \leqslant \gamma_i^2 \quad \text{for } i \in M, \quad u_i^1 \geqslant u_i^2,\ \gamma_i^1 \geqslant \gamma_i^2 \quad \text{for } i \notin M \tag{14}$$

and

$$(\mathscr{A}^{\mathrm{t}}\eta)_i \geqslant 0,\ (\mathscr{B}^{\mathrm{t}}\eta)_i > 0 \quad \text{for } i \in M, \quad (\mathscr{A}^{\mathrm{t}}\eta)_i \leqslant 0,\ (\mathscr{B}^{\mathrm{t}}\eta)_i \leqslant 0 \quad \text{for } i \notin M. \tag{15}$$

Inequalities (15) follow from diagonal dominance of matrices $\mathscr{A}^{\mathrm{t}}$ and $\mathscr{B}^{\mathrm{t}}$ (strict for $\mathscr{B}^{\mathrm{t}}$) and from the nonpositiveness for off-diagonal entries of these matrices.

Multiplying Eq. (11), written for F^1 and F^2, by η, we obtain

$$(u^1 - u^2, \mathscr{A}^{\mathrm{t}}\eta) + (\gamma^1 - \gamma^2, \mathscr{B}^{\mathrm{t}}\eta) + (C(u^1) - C(u^2), \eta) = (F^1 - F^2, \eta). \tag{16}$$

By the definition of η the right-hand side in (16) is nonnegative. On the other hand, if we suppose that $J_- \neq \emptyset$, then from (14) and (15) we deduce that left-hand side in (16) is negative. So, $J_- = \emptyset$ and $M = I_-$.

Let us suppose now that $I_- \neq \emptyset$. Then from (16) we get

$$0 \leqslant (F^1 - F^2, \eta) \leqslant (u^1 - u^2, \mathscr{A}^{\mathrm{t}}\eta) \leqslant ((u^1 - u^2)_{I_-}, \mathscr{A}^{\mathrm{t}}_{I_-I_-}\eta_{I_-}), \tag{17}$$

where $\mathscr{A}^{\mathrm{t}}_{I_-I_-}$ is the $I_- \times I_-$ submatrix of $\mathscr{A}^{\mathrm{t}}$. The matrix $\mathscr{A}^{\mathrm{t}}_{I_-I_-}$, being a submatrix of matrix $\mathscr{A}^{\mathrm{t}}$, is an M-matrix with weak diagonal dominance in columns. Also, as $\eta_{I_-} \equiv 1$, the coordinates of the vector $\mathscr{A}^{\mathrm{t}}_{I_-I_-}\eta_{I_-}$ are nonnegative and at least one of them is positive, i.e., the right-hand side of (17) is negative if $I_- \neq \emptyset$. We get again the contradiction that proves the desired inequality $(u^1, \gamma^1) \geqslant (u^2, \gamma^2)$.

Obviously, the second statement of theorem implies the uniqueness of a solution for problem (11). □

Let $\mathscr{A} = A$, $\mathscr{B} = B$. Then properties (5)–(10) ensure the validity of all assumptions of Theorem 1 for algebraic problem (4). Thus, the following statement holds:

Theorem 2. *The implicit mesh scheme* (2) *and the semi-implicit mesh scheme* (3) *have the unique solutions.*

4. Iterative methods

In this section, we study the convergence of asynchronous iterative methods from Introduction, which appear as the two-stage additive Schwarz alternating methods (ASAMs) for solving problem (4). Then we prove the comparison result which implies that these methods have to be convergent not slower than usual point Jacobi iterative method. Last, we derive a geometric rate of convergence for Jacobi method, for asynchronous iterative methods as well.

For simplicity, without loss of generality, we suppose that the domain Ω is decomposed into two overlapping subdomains Ω_1 and Ω_2, consisting of the elements of triangulation T_h. We arrange the nodes of the mesh as follows. First, we enumerate the nodes lying in the nonoverlapping part of the first subdomain, (namely $x \in (\bar{\Omega}_1 \backslash \bar{\Gamma}_1) \backslash \overline{\Omega_1 \cap \Omega_2}$), then the nodes in the overlapping zone $x \in \overline{\Omega_1 \cap \Omega_2} \backslash \bar{\Gamma}_1$ and at last the nodes in the nonoverlapping part of the second subdomain. A vector $u \in \mathbb{R}^{N_0}$, $u \Leftrightarrow u_h(x)$, takes the form $u = (u_{11}, u_{12}, u_{22})^t$ with subvectors u_{ij} corresponding to enumeration of the nodes.

This decomposition implies also the partitioning of the matrices and nonlinear operators

$$A = (A_{ij})_{ij=1}^3, \quad B = (B_{ij})_{ij=1}^3, \quad C = \mathrm{diag}(C_1, C_2, C_3).$$

Note that $A_{ij} \leqslant 0$, $B_{ij} \leqslant 0$ for $i \neq j$ and the blocks $A_{13}, A_{31}, B_{13}, B_{31}$ are equal to zero.

We use also the following notations:

$$A_0^1 = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \quad B_0^1 = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}, \quad A_1^1 = \mathrm{diag}(0, A_{23}), \quad B_1^1 = \mathrm{diag}(0, B_{23}),$$

$$A_0^2 = \begin{pmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{pmatrix}, \quad B_0^2 = \begin{pmatrix} B_{22} & B_{23} \\ B_{32} & B_{33} \end{pmatrix}, \quad A_1^2 = \mathrm{diag}(A_{21}, 0), \quad B_1^2 = \mathrm{diag}(B_{21}, 0),$$

$$C^1 = \mathrm{diag}(C_1, C_2), \quad C^2 = \mathrm{diag}(C_2, C_3).$$

Let further

$$u_1 = (u_{11}, u_{12})^t, \quad u_2 = (u_{12}, u_{22})^t$$

and similar for all other vectors.

Then ASAM has the form (18), (19)

$$\begin{aligned} &A_0^1 v_1^{k+1} + B_0^1 \eta_1^{k+1} + C^1 v_1^{k+1} = F_1 - A_1^1 u_2^k - B_1^1 \gamma_2^k, \eta_1^{k+1} \in H(v_1^{k+1}), \\ &A_0^2 w_2^{k+1} + B_0^2 \xi_2^{k+1} + C^2 w_2^{k+1} = F_2 - A_1^2 u_1^k - B_1^2 \gamma_1^k, \xi_2^{k+1} \in H(w_2^{k+1}), \end{aligned} \tag{18}$$

$$\begin{aligned} &u_{11}^{k+1} = v_{11}^{k+1},\ u_{22}^{k+1} = w_{22}^{k+1},\ u_{12}^{k+1} = \alpha v_{12}^{k+1} + (1-\alpha) w_{12}^{k+1}, \\ &\gamma_{11}^{k+1} = \eta_{11}^{k+1},\ \gamma_{22}^{k+1} = \xi_{22}^{k+1},\ \gamma_{12}^{k+1} = \alpha \eta_{12}^{k+1} + (1-\alpha) \xi_{12}^{k+1}, \end{aligned} \tag{19}$$

with an initial guess (u^0, γ^0) and $\alpha \in (0, 1)$.

Let now every subproblem in (18) be solved by using a finite number of iterations of an inner iterative algorithm. Then we derive a two-stage ASAM. Below we write two-stage ASAMs, corresponding to ASM1 and ASM2 from Introduction.

Let for $i = 1, 2$

$$A_0^i = M_i + N_i, \quad B_0^i = K_i + L_i$$

be the regular splittings of A and B with $\mathrm{diag}(A_0^i) \subseteq M_i$, $\mathrm{diag}(B_0^i) \subseteq K_i$ and $N_i \leqslant 0, L_i \leqslant 0$. Starting from the initial guess

$$z_{1,0} = u_1^k, \quad z_{2,0} = u_2^k, \quad \varepsilon_{1,0} = \gamma_1^k, \quad \varepsilon_{2,0} = \gamma_2^k,$$

we solve the subproblems in (18) by the following iterative methods:

$$M_1 z_{1,i} + K_1 \varepsilon_{1,i} + C^1 z_{1,i} = \varphi_1^k - N_1 z_{1,i-1} - L_1 \varepsilon_{1,i-1},$$
$$\varepsilon_{1,i} \in H(z_{1,i}), \quad i = 1, \dots, p_1, \tag{20}$$

$$M_2 z_{2,i} + K_2 \varepsilon_{2,i} + C^2 z_{2,i} = \varphi_2^k - N_1 z_{2,i-1} - L_1 \varepsilon_{2,i-1},$$
$$\varepsilon_{2,i} \in H(z_{2,i}), \quad i = 1, \dots, p_2, \tag{21}$$

set

$$v_1^{k+1} \equiv z_{1,p_1}, \quad \eta_1^{k+1} \equiv \varepsilon_{1,p_1}, \quad w_2^{k+1} \equiv z_{2,p_2}, \quad \xi_2^{k+1} \equiv \varepsilon_{2,p_2}$$

and then update the outer iterations using formulas (19).

Here

$$\varphi_1^k = F_1 - A_1^1 u_2^k - B_1^1 \gamma_2^k, \quad \varphi_2^k = F_2 - A_1^2 u_1^k - B_1^2 \gamma_1^k$$

for method ASM1, when we calculate all subproblems by using inner iterative methods until we reach the desired accuracy in all subproblems. After that, we send the calculated $v_1^{k+1}, w_2^{k+1}, \eta_1^{k+1}, \xi_2^{k+1}$ to master processor to update the outer iterations by using formulas (19). On the other hand, for method ASM2 the formulas for φ_i^k are changed by

$$\varphi_1^k = F_1 - A_1^1 w_2^{k+1} - B_1^1 \xi_2^{k+1}$$

or by

$$\varphi_2^k = F_2 - A_1^2 v_1^{k+1} - B_1^2 \eta_1^{k+1},$$

depending on which of the subproblems was solved faster.

Along with these methods we also consider Jacobi method (JM). Let $A^0 = \operatorname{diag}(A)$, $A^1 = A - A^0$ and $B^0 = \operatorname{diag}(B)$, $B^1 = B - B^0$. Then the iterative method JM has the form

$$A^0 u_J^{k+1} + B^0 \gamma_J^{k+1} + C u_J^{k+1} = F - A^1 u_J^k - B^1 \gamma_J^k, \gamma_J^{k+1} \in H(u_J^{k+1}). \tag{22}$$

The proof of the following statement is simple and direct.

Lemma 2. *The matrices in all iterative methods* (18), (20), (21), (22) *inherit the properties of the matrices A and B, namely*:

$$A^0, A_0^i, M_i, \; i = 1, 2, \text{ are weakly diagonally dominant in columns } M\text{-matrices},$$

$$B^0, B_0^i, K_i, \; i = 1, 2, \text{ are strictly diagonally dominant in columns } M\text{-matrices}.$$

Using Theorem 1 (part 2), we can also prove the following:

Lemma 3. *The pairs $(\bar{u}, \bar{\gamma})$ and $(\underline{u}, \underline{\gamma})$ are, respectively, a supersolution and a subsolution for all problems* (18), (20), (21), (22).

Owing to Lemmas 2 and 3 and to Theorem 1 (part 1), all iterative methods (18)–(19), and (20), (21), (19), and (22) are correctly defined.

Below, in Lemmas 4 and 5, we use the second statement of Theorem 1 to prove the comparison results.

Lemma 4. *Let* $(\bar{u}_J^k, \bar{\gamma}_J^k)$ *and* $(\underline{u}_J^k, \underline{\gamma}_J^k)$ *be the kth iterations of JB* (22) *with the initial guesses* $(\bar{u}_J^0, \bar{\gamma}_J^0) = (\bar{u}, \bar{\gamma})$ *and* $(\underline{u}_J^0, \underline{\gamma}_J^0) = (\underline{u}, \underline{\gamma})$, *respectively. Let also* (u, γ) *be the exact solution to problem* (4). *Then*

$$(\bar{u}_J^k, \bar{\gamma}_J^k) \geqslant (\bar{u}_J^{k+1}, \bar{\gamma}_J^{k+1}) \geqslant (u, \gamma), \tag{23}$$

$$(\underline{u}_J^k, \underline{\gamma}_J^k) \leqslant (\underline{u}_J^{k+1}, \underline{\gamma}_J^{k+1}) \leqslant (u, \gamma). \tag{24}$$

Proof. We prove by induction inequalities (23). First, as $(\bar{u}_J^0, \bar{\gamma}_J^0) = (\bar{u}, \bar{\gamma})$, then

$$A^0 \bar{u}_J^1 + B^0 \bar{\gamma}_J^1 + C \bar{u}_J^1 = F - A^1 \bar{u}_J^0 - B^1 \bar{\gamma}_J^0 \leqslant A^0 \bar{u}_J^0 + B^0 \bar{\gamma}_J^0 + C \bar{u}_J^0,$$

which implies $(\bar{u}_J^0, \bar{\gamma}_J^0) \geqslant (\bar{u}_J^1, \bar{\gamma}_J^1)$ due to Theorem 1 (part 2).

For any $k \geqslant 1$ we have

$$A^0 \bar{u}_J^{k+1} + B^0 \bar{\gamma}_J^{k+1} + C \bar{u}_J^{k+1} = F - A^1 \bar{u}_J^k - B^1 \bar{\gamma}_J^k$$
$$\leqslant F - A^1 \bar{u}_J^{k-1} - B^1 \bar{\gamma}_J^{k-1} = A^0 u_J^k + B^0 \bar{\gamma}_J^k + C \bar{u}_J^k,$$

whence $(\bar{u}_J^{k+1}, \bar{\gamma}_J^{k+1}) \leqslant (\bar{u}_J^k, \bar{\gamma}_J^k)$.

The inequality $(\bar{u}_J^{k+1}, \bar{\gamma}_J^{k+1}) \geqslant (u, \gamma)$, we prove also by induction on $k = 0, 1, \ldots$ and by using the relations

$$A^0 \bar{u}_J^{k+1} + B^0 \bar{\gamma}_J^{k+1} + C \bar{u}_J^{k+1} = F - A^1 \bar{u}_J^k - B^1 \bar{\gamma}_J^k \geqslant F - A^1 u - B^1 \gamma = A^0 u + B^0 \gamma + Cu. \qquad \square$$

Now we prove a comparison result for the iterations of two-stage Schwarz method (20)–(21), (19) and JM (22) when the last starts from a subsolution or a supersolution.

Lemma 5. *Let* $(\bar{u}_J^k, \bar{\gamma}_J^k)$ *and* $(\underline{u}_J^k, \underline{\gamma}_J^k)$ *be the kth iterations of JM* (22) *with the initial guesses* $(\bar{u}_J^0, \bar{\gamma}_J^0) = (\bar{u}, \bar{\gamma})$ *and* $(\underline{u}_J^0, \underline{\gamma}_J^0) = (\underline{u}, \underline{\gamma})$, *respectively. Let further* (u^k, γ^k) *be a kth iteration of method* (20)–(21), (19) *with an initial guess* $(u^0, \gamma^0) \in \langle (\underline{u}, \underline{\gamma}), (\bar{u}, \bar{\gamma}) \rangle$. *Then*

$$(\underline{u}_J^k, \underline{\gamma}_J^k) \leqslant (u^k, \gamma^k) \leqslant (\bar{u}_J^k, \bar{\gamma}_J^k). \tag{25}$$

Proof. We prove the inequality $(u^k, \gamma^k) \leqslant (\bar{u}_J^k, \bar{\gamma}_J^k)$ proceeding by induction. First, this inequality is valid for $k = 0$ because of the definition of the initial guess. Now we prove that

$$z_{1,i} \leqslant \bar{u}_{J,1}^1, \quad \varepsilon_{1,i} \leqslant \bar{\gamma}_{J,1}^1 \quad \text{and} \quad z_{2,i} \leqslant \bar{u}_{J,2}^1, \quad \varepsilon_{2,i} \leqslant \bar{\gamma}_{J,2}^1 \quad \forall i \geqslant 1. \tag{26}$$

We denote by $A_J^0 = \mathrm{diag}(A_0^1)$, $B_J^0 = \mathrm{diag}(B_0^1)$ and by $A_J^1 = A_0^1 - A_J^0$, $B_J^1 = B_0^1 - B_J^0$.

For $i = 0$ we have $z_{1,0} = u_1^0, \varepsilon_{1,0} = \gamma_1^0$, $z_{2,0} = u_2^0, \varepsilon_{2,0} = \gamma_2^0$, whose coordinates are less than or equal to the corresponding coordinates of $\bar{u}_J^0, \bar{\gamma}_J^0$.

Now we prove these inequalities for $i=1$. From Eq. (22) we derive

$$A^0_J\bar{u}^1_{J,1} + B^0_J\bar{\gamma}^1_{J,1} + C^1\bar{u}^1_{J,1} = F_1 - A^1_J\bar{u}^0_{J,1} - B^1_J\bar{\gamma}^0_{J,1} - A^1_1\bar{u}^0_{J,2} - B^1_1\bar{\gamma}^0_{J,2}.$$

From this relation, Eq. (20) and the inequalities for matrices $N_1 \leqslant 0, L_1 \leqslant 0, A^1_1 \leqslant 0, B^1_1 \leqslant 0, A^1_J - N_1 \leqslant 0, B^1_J - L_1 \leqslant 0$ and for vectors (see Lemma 4) $\bar{u}^1_{J,1} \leqslant \bar{u}^0_{J,1}, \bar{\gamma}^1_{J,1} \leqslant \bar{\gamma}^0_{J,1}$, we have

$$\begin{aligned}
M_1 z_{1,1} + K_1\varepsilon_{1,1} + C^1 z_{1,1} &= F_1 - A^1_1 u^0_2 - B^1_1\gamma^0_2 - N_1 z_{1,0} - L_1\varepsilon_{1,0}\\
&\leqslant F_1 - A^1_1\bar{u}^0_{J,2} - B^1_1\bar{\gamma}^0_{J,2} - N_1\bar{u}^0_{J,1} - L_1\bar{\gamma}^0_{J,1}\\
&= A^0_J\bar{u}^1_{J,1} + (A^1_J - N_1)\bar{u}^0_{J,1} + B^0_J\bar{\gamma}^1_{J,1} + (B^1_J - L_1)\bar{\gamma}^0_{J,1} + C^1\bar{u}^1_{J,1}\\
&\leqslant A^0_J\bar{u}^1_{J,1} + (A^1_J - N_1)\bar{u}^1_{J,1} + B^0_J\bar{\gamma}^1_{J,1} + (B^1_J - L_1)\bar{\gamma}^1_{J,1} + C^1\bar{u}^1_{J,1}\\
&= M_1\bar{u}^1_{J,1} + K_1\bar{\gamma}^1_{J,1} + C^1\bar{u}^1_{J,1}.
\end{aligned}$$

As $\varepsilon_{1,1} \in H(z_{1,1})$ and $\bar{\gamma}^1_{J,1} \in H(\bar{u}^1_{J,1})$, we can apply Theorem 1 (part 2) with $\mathscr{A} = M_1$, $\mathscr{B} = K_1$ and obtain

$$z_{1,1} \leqslant \bar{u}^1_{J,1}, \quad \varepsilon_{1,1} \leqslant \bar{\gamma}^1_{J,1}.$$

Using the same procedure we prove that $z_{2,1} \leqslant \bar{u}^1_{J,2}$, $\varepsilon_{2,1} \leqslant \bar{\gamma}^1_{J,2}$ and then, sequently, inequalities (26) for all i. Thus, we obtain $v^1_1 \leqslant \bar{u}^1_J$, $w^1_2 \leqslant \bar{u}^1_{J,2}$, $\eta^1_1 \leqslant \bar{\gamma}^1_{J,1}$, $\xi^1_2 \leqslant \bar{\gamma}^1_{J,2}$ and, as a consequence, $(u^1,\gamma^1) \leqslant (\bar{u}^1_J, \bar{\gamma}^1_J)$. Further by induction we prove that $(u^k,\gamma^k) \leqslant (\bar{u}^k_J, \bar{\gamma}^k_J)$. □

Lemma 6. *Let assumptions* (1), (5)–(8) *be valid and* (u,γ) *be the exact solution to problem* (4). *Then for JM's iterations the following estimates hold*:

$$\|A^0(\bar{u}^{k+1}_J - u) + B^0(\bar{\gamma}^{k+1}_J - \gamma)\|_1 \leqslant q\|A^0(\bar{u}^k_J - u) + B^0(\bar{\gamma}^k_J - \gamma)\|_1, \tag{27}$$

$$\|A^0(\underline{u}^{k+1}_J - u) + B^0(\underline{\gamma}^{k+1}_J - \gamma)\|_1 \leqslant q\|A^0(\underline{u}^k_J - u) + B^0(\underline{\gamma}^k_J - \gamma)\|_1 \tag{28}$$

with $q = (c_{AB} + \alpha\beta)/(c_{AB} + \alpha) < 1$, $c_{AB} = \max_{1 \leqslant i \leqslant N_0} a_{ii}/b_{ii}$.

Here $\|v\|_1 = \sum_{i=1}^{N_0} |v_i|$ *and* $c_{AB} = 2\tau(1 + h_2^2/h_1^2)/h_2(\tau + h_2)$ *for the implicit scheme, while* $c_{AB} = 2\tau(1 + h_2^2/h_1^2)/h_2^2$ *for the semi-implicit scheme.*

Proof. Let us prove the first estimate, when the initial guess $(u^0_J, \gamma^0_J) = (\bar{u}, \bar{\gamma})$.

Using the notations $P_1 = -A^1(A^0)^{-1}$, $P_2 = -B^1(B^0)^{-1}$, we rewrite equation (22) as

$$A^0\bar{u}^{k+1}_J + B^0\bar{\gamma}^{k+1}_J + C\bar{u}^{k+1}_J = P_1 A^0\bar{u}^k_J + P_2 B^0\bar{\gamma}^k_J + F. \tag{29}$$

Because of (5)–(7) all entries of matrices P_1, P_2 are nonnegative and

$$\|P_1\|_1 \leqslant 1, \quad \|P_2\|_1 = \beta \leqslant 1.$$

We recall that owing to Lemma 4

$$(\bar{u}^k_J, \bar{\gamma}^k_J) \geqslant (\bar{u}^{k+1}_J, \bar{\gamma}^{k+1}_J) \geqslant (u,\gamma).$$

From Eq. (29) we obtain are inequality relating errors $z^k (= A^0(u^k - u) \geqslant 0)$ and $\eta^k (= B^0(\gamma^k - \gamma) \geqslant 0)$:

$$z^{k+1} + \eta^{k+1} \leqslant P_1 z^k + P_2\eta^k. \tag{30}$$

Let us denote by p_i^1, p_i^2 the sums of all entries of ith column for the matrices P_1, P_2, respectively. Let also $w^k = z^k + \eta^k$, $c_i^k = z_i^k/\eta_i^k$.

Note, that $0 \leqslant c_i^k \leqslant a_{ii}/b_{ii}\alpha$. From (30) we deduce

$$\|w^{k+1}\|_1 \leqslant \sum_{i=1}^{N_0} (p_i^1 z_i^k + p_i^2 \eta_i^k).$$

Because $\eta_i^k = [1/(1+c_i^k)]w_i^k$, $z_i^k = [c_i^k/(1+c_i^k)]w_i^k$,

$$\|w^{k+1}\|_1 \leqslant \sum_{i=1}^{N_0} \frac{p_i^2 + p_i^1 c_i^k}{1+c_i^k} w_i^k.$$

As all the members in sum of this inequality are nonnegative, we obtain

$$\|w^{k+1}\|_1 \leqslant \max_i \frac{p_i^2 + p_i^1 c_i^k}{1+c_i^k} \|w^k\|_1.$$

Direct calculations lead to the estimates

$$\max_i \frac{p_i^2 + p_i^1 c_i^k}{1+c_i^k} \leqslant \max_i \frac{\beta + c_i^k}{1+c_i^k} = \frac{\beta + \max_i c_i^k}{1+\max_i c_i^k} \leqslant \frac{\alpha\beta + \max_i \frac{a_{ii}}{b_{ii}}}{\alpha + \max_i \frac{a_{ii}}{b_{ii}}} = q$$

and inequality (27) is proved. The proof of (28) is similar. The value of c_{AB} depends on mesh parameters τ. h_1, h_2 can be easily calculated, because $a_{ii} = 2/h_1^2 + 2/h_2^2$, while $b_{ii} = 1/\tau + 1/h_2$ for the implicit scheme and $b_{ii} = 1/\tau$ for the semi-implicit scheme. $\square$

As a straight consequence of Lemmas 5 and 6 we derive the following:

Theorem 3. *Iterative method* (20)–(21), (19) *with an initial guess* $(u^0, \gamma^0) \in \langle(\underline{u}, \underline{\gamma}), (\bar{u}, \bar{\gamma})\rangle$ *converges with geometric rate of convergence*:

$$\|A^0(u^{k+1} - u) + B^0(\gamma^{k+1} - \gamma)\|_1 \leqslant q\|A^0(u^k - u) + B^0(\gamma^k - \gamma)\|_1, \tag{31}$$

where $q < 1$ *is defined in Lemma* 6.

5. Numerical results

To validate theoretical results, the following numerical example was considered. Let $\Omega =]0,1[\times]0,1[$ with the boundary Γ divided in two parts such that $\Gamma_D = \{x \in \partial\Omega : x_2 = 0 \vee x_2 = 1\}$ and $\Gamma_N = \Gamma \backslash \Gamma_D$. Moreover let $T = 1$. Let us consider the case where the phase change temperature $u_{SL} = 1$, the latent heat $L = 1$ and the density $\rho = 1$. Let the velocity be $v(t) = \frac{1}{5}$. Our numerical example is

$$\frac{\partial H}{\partial t} - \Delta K + v(t)\frac{\partial H}{\partial x_2} = f(x;t) \quad \text{on } \Omega,$$

$$u(x_1, x_2; t) = (x_1 - \tfrac{1}{2})^2 + (x_2 - \tfrac{1}{2})^2 - \tfrac{1}{2}\,e^{-4t} + \tfrac{5}{4} \quad \text{on } \Gamma_D,$$

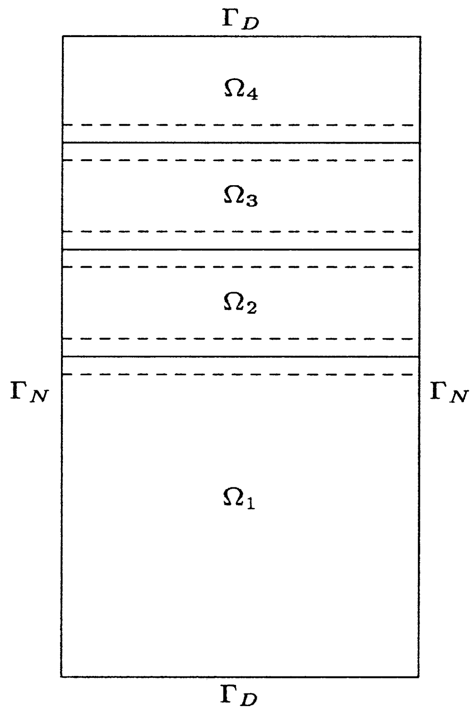

Fig. 1. The decomposition used in model continuous casting problem.

$$\frac{\partial u}{\partial n} = 1 \quad \text{on } \Gamma_{\mathrm{N}},$$

$$u(x_1, x_2; 0) = (x_1 - \tfrac{1}{2})^2 + (x_2 - \tfrac{1}{2})^2 + \tfrac{1}{2} \quad \text{on } \Omega,$$

where

$$K(u) = \begin{cases} u & \text{if } u < u_{\mathrm{SL}}, \\ 2u - 1 & \text{if } u \geqslant u_{\mathrm{SL}} \end{cases}$$

and

$$H(u) = \begin{cases} 2u & \text{if } u < u_{\mathrm{SL}}, \\ [2u_{\mathrm{SL}}, 2u_{\mathrm{SL}} + \rho L] & \text{if } u = u_{\mathrm{SL}}, \\ 6u - 4u_{\mathrm{SL}} + \rho L & \text{if } u > u_{\mathrm{SL}}. \end{cases}$$

Furthermore,

$$f(x; t) = \begin{cases} 4\mathrm{e}^{-4t} + \frac{1}{5}(4x_2 - 2) - 4 & \text{if } u < u_{\mathrm{SL}}, \\ 12\mathrm{e}^{-4t} + \frac{1}{5}(12x_2 - 6) - 8 & \text{if } u \geqslant u_{\mathrm{SL}}. \end{cases}$$

The stopping criterion of the outer iterations was the value of the L_2-norm of residual $\|r\|_{L_2} = \|Au + B\gamma + \delta - f\|_{L_2} \leqslant 10^{-3}$. We use through all the calculations the decomposition presented in Fig. 1. The subdomain Ω_1 is roughly twice as big as other subdomains.

Table 1
The number of outer iterations and calculation times in seconds for different grids for 4 processors

Grid	Over	ASM1 ♯ iterations	ASM1 T(s)	ASM2 ♯ iterations	ASM2 T(s)
$65 \times 65 \times 128$	4	17	14.8	8	11.4
$129 \times 129 \times 256$	8	16	92.1	11	73.0
$257 \times 257 \times 512$	16	19	1184	17	1120

5.1. Implicit scheme

In our first test case, we changed the number of grid points both in time and in space. We solved the problem by using implicit scheme (2). The results can be seen in Table 1. The over is the number of grid lines in the overlapping area. The inner iterations was performed till all of the processors have reached the desired accuracy $\|r\|_{L_2} \leqslant 10^{-3}$. Due to this the number of inner iterations can be different for different processors.

In our second case, we tested the geometrical convergence of ASM1 and ASM2 with implicit scheme. The calculation grid was $65 \times 65 \times 128$. The size of the overlapping was four grid lines. In this case, the parameter α in Eq. (1) was $\alpha = 6$. To reach the accuracy $\|r\|_{L_2} \leqslant 10^{-3}$, we need 15

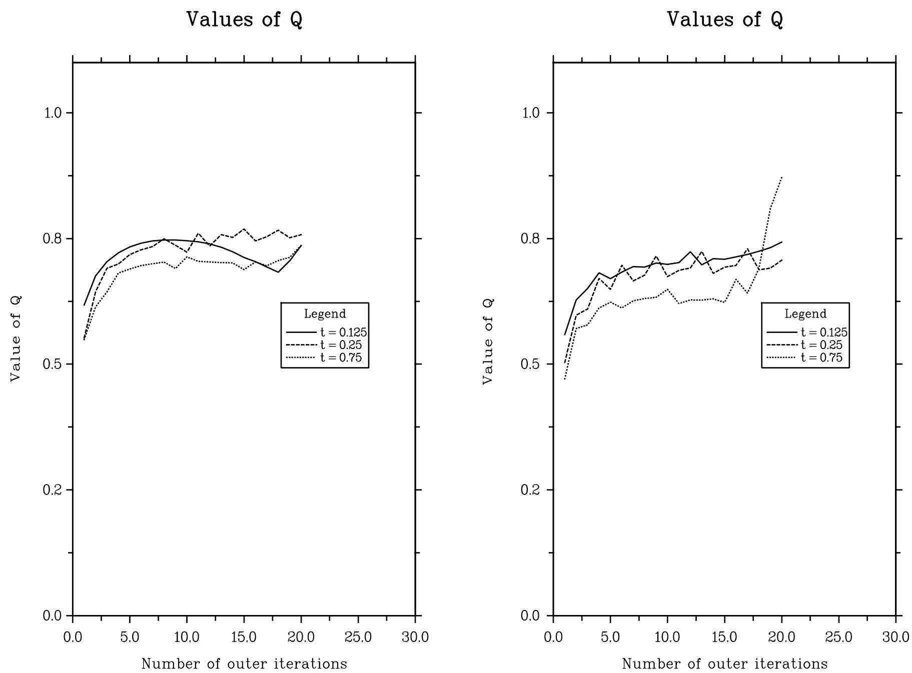

Fig. 2. The behavior of ratio q for the method ASM1 (on the left) and for the method ASM2 (on the right) for three different fixed time levels.

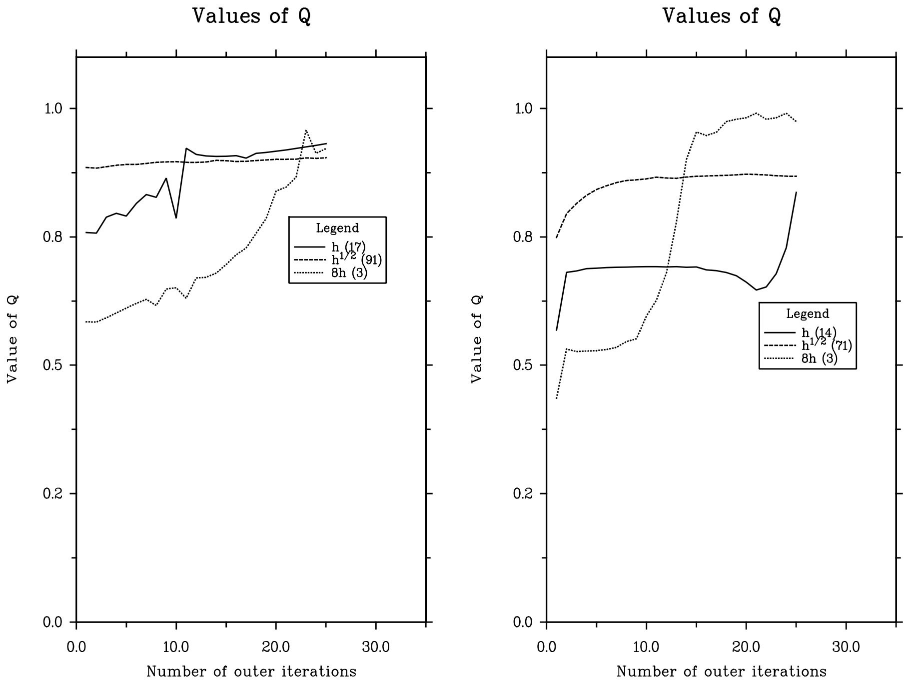

Fig. 3. The behavior of ratio q for the method ASM1 (on the left) and for the method ASM2 (on the right) when the time discretization parameter τ is changed.

outer iterations. We keep calculating further to see how the parameter q is changing. In Fig. 2 on the left is plotted the values of q in three different time levels for method ASM1 and on the right for the method ASM2.

In our third case, we tested the effects of parameter τ for the asynchronous additive Schwarz method with implicit scheme. In all of these figures, we look to the situation in the middle point of time interval $t = 0.5$. First, we fixed the number of grid points in space and changed the number of time steps. In Fig. 3, on the left, the behavior of the parameter q for ASM1 method with implicit scheme can be seen and, on the right, the behavior for method ASM2.

It can be seen from Fig. 3 that when the time step is decreased the q gets smaller as is expected from the theoretical results (Lemma 6). It can also be seen that after some number of iterations the values of q get worse. The number in the brackets after the parameter h is just the number of outer iterations needed to solve the problem with the wanted accuracy.

In the fourth test case, we fixed the calculation grid both for space and time. We had 65×65 grid points in space and 128 time steps. We changed the parameter α.

The values of parameter q for the method ASM1 can be seen in Fig. 4 on the left and for the method ASM2 on the right. The numbers on the brackets are the number of outer iterations to reach the desired accuracy.

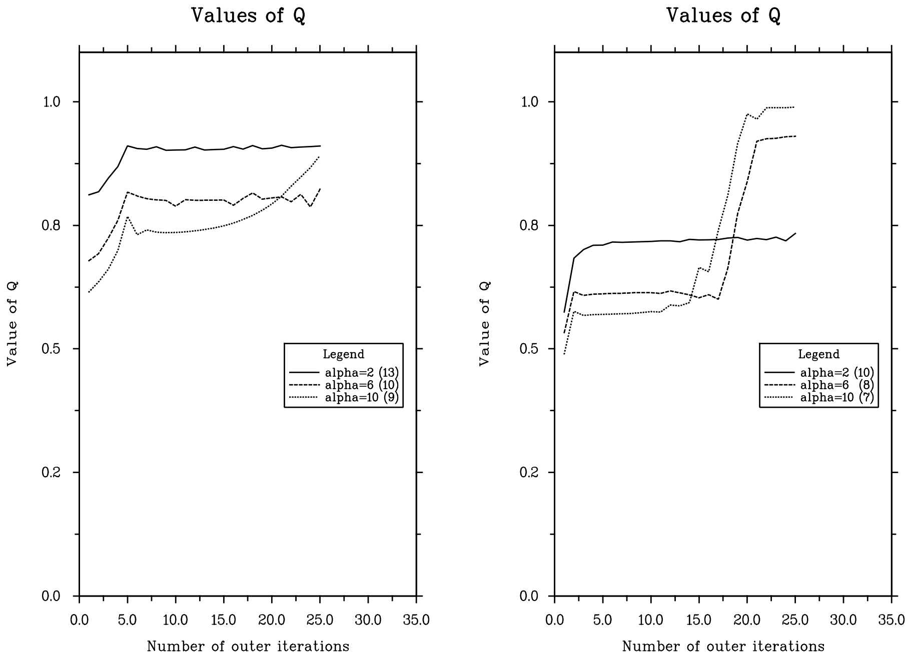

Fig. 4. The behavior of ratio q for the method ASM1 (on the left) and ASM2 (on the right) when the parameter α is changed.

Table 2
The number of outer iterations and calculation times in seconds for different grids for 4 processors

Grid	Over	ASM1 ♯ iterations	ASM1 T(s)	ASM2 ♯ iterations	ASM2 T(s)
$65 \times 65 \times 128$	4	17	12.2	13	10.6
$129 \times 129 \times 256$	8	16	84.5	14	65.9
$257 \times 257 \times 512$	16	19	1171	17	1056

5.2. Semi-implicit scheme

We solve the same problem like for implicit scheme to compare these methods against each other. In our first test case, we changed the number of grid points both in time and in space. We solved the problem by using semi-implicit scheme (3). The results can be seen Table 2. The over is the number of grid lines in the overlapping area. The inner iterations was performed till all of the processors have reached the desired accuracy $\|r\|_{L_2} \leqslant 10^{-3}$. Due to this, the number of inner iterations can be different for different processors.

In our second case, we tested the geometrical convergence of ASM1 and ASM2 with semi-implicit scheme. The calculation grid was $65 \times 65 \times 128$. The size of the overlapping was 4 grid lines.

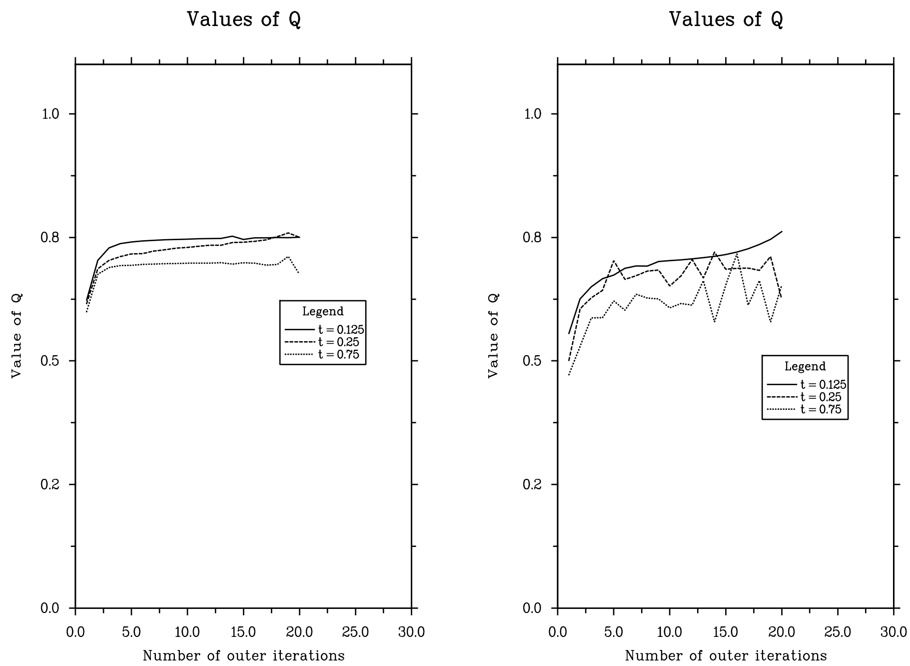

Fig. 5. The behavior of ratio q for the method ASM1 (on the left) and for the method ASM2 (on the right) for three different fixed time levels.

The parameter $\alpha = 6$. To reach the accuracy $\|r\|_{L_2} \leqslant 10^{-3}$, we need 15 outer iterations. We keep calculating further to see how the parameter q is changing. In Fig. 5 on the left is plotted the values of q in three different time level for method ASM1 and on the right for the method ASM2.

In our third case, we tested the effect of parameter τ for the asynchronous additive Schwarz method with semi-implicit scheme. In all of these figures, we look at the situation in the middle point of time interval $t = 0.5$. First we fixed the number of grid points in space and changed the number of time steps. In Fig. 6, on the left, the behavior of the parameter q for ASM1 method with characteristic scheme can be seen and on the right, the behavior for the method ASM2.

It can be seen from Fig. 6 that when the time step is decreased the q gets smaller as is expected from the theoretical results (Lemma 6). It can also be seen that after some number of iterations the values of q get worse similar to the case of implicit scheme. The number in the brackets after the parameter h is just the number of outer iterations needed to solve the problem with the wanted accuracy.

In the fourth test case, we fixed the calculation grid both for space and time. We had 65×65 grid points in space and 128 time steps. We changed the parameter α.

The values of parameter q for the method ASM1 can be seen in Fig. 7 on the left and for the method ASM2 on the right. The numbers in the brackets are the number of outer iterations to reach the desired accuracy.

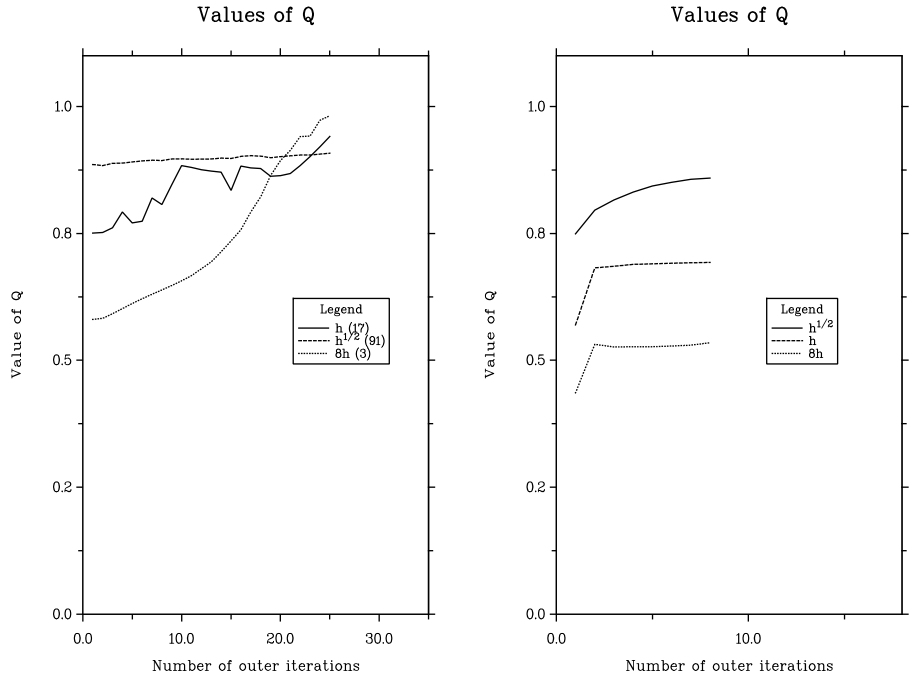

Fig. 6. The behavior of ratio q for the method ASM1 (on the left) and ASM2 (on the right) when the time discretization parameter τ is changed.

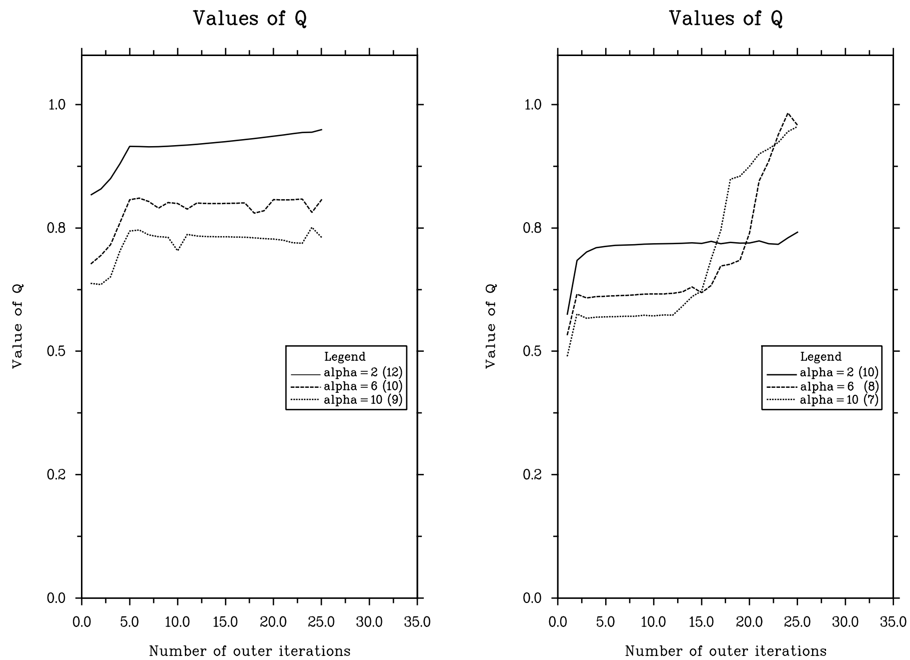

Fig. 7. The behavior of ratio q for the method ASM1 (on the right) and for the method ASM2 (on the left) when the parameter α is changed.

6. Conclusions

Two mesh schemes with two different kind of discretizations for convection term was considered, implicit and semi-implicit scheme. The model problem was solved by using both asynchronous methods ASM1 and ASM2. It can be seen from Tables 1 and 2 that ASM2 takes less outer iterations than ASM1 and is thus faster of these two methods. From these tables, it also can be seen that semi-implicit scheme and implicit scheme take approximately same number of outer iterations, but semi-implicit scheme is slightly faster than implicit scheme.

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